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VALUES OF NON-ATOMIC GAMES, IV: THE VALUE AND THE CORE

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PREFACE

This is a direct continuation of RM-5468-PR, RM-5842-PR, and RM-6216, subtitled "Part I: The Axiomatic Approach," "Part II: The Random Order Approach," and "Part III: Values and Derivatives," respectively. Non-atomic games are models for competitive situations in which there are many participants, no one of whom has any appreciable influence as an individual. Such games have recently attracted attention as models for mass phenomena in economics.

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Part of this work was supported by USAF Project Rand. Dr. Aumann is a professor of mathematics at the Hebrew University of Jerusalem, and a Rand consultant.

SUMMARY

The <u>value</u> of an n-person game is a function that associates to each player a number that, intuitively speaking, represents an a priori opinion of what it is worth to him to play in the game. A <u>non-atomic</u> game is a special kind of infinite-person game, in which no individual player has significance; such games have recently attracted attention as models for mass phenomena in economics. This is the fourth in a series of Rand Memoranda in which the value concept, originally defined only for finite-person games, is extended to non-atomic games.

In this Memorandum, the relationship between the value and another solution concept is developed. The <u>core</u> of a game is the set of outcomes that, intuitively speaking, no coalition of players can improve upon. The core is a basic concept that has been studied widely by both game theorists and economists. The main object of the present paper is to prove that under suitable assumptions, the core of a non-atomic game consists of a single outcome, and that this outcome coincides with the value. The assumptions [superadditivity and homogeneity of degree 1] are satisfied in many cases of interest, including a basic economic application.

CONTENTS

VALUES OF NON-ATOMIC GAMES
PART IV: THE VALUE AND THE CORE

26. INTRODUCTION TO PART IV, AND STATEMENT OF RESULTS

This is the fourth in a series of Rand Memoranda with the overall title "Values of non-atomic games"*. Familiarity with the previous parts will be assumed throughout. Numeration of the sections will be continued serially here, to enable easy reference to the previous parts. Other conventions established previously will also be maintained here.

The main object of Part IV is the proof of

THEOREM F. *Let* v *be a superadditive set function in* pNA *that is homogeneous of degree* 1. *Then the core of* v *has a unique member, which coincides with the value of* v.

Several of the terms used in the statement of Theorem F may not be familiar to the reader. A set function v is *superadditive* if for disjoint S and T,

$$v(S \cup T) \geq v(S) + v(T).$$

It is *homogeneous of degree* 1 if

$$v^*(\alpha\chi_S) = \alpha v(S)$$

for all α in $[0,1]$ and all sets $S \in \mathcal{C}$, where v^* is the

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*For the previous parts, see [I, II, III] in the list of references.

extension defined by Theorem D (see Secs. 21 and 22). An equivalent* formulation of this homogeneity condition is

$$v^*(\alpha f) = \alpha v^*(f)$$

for all α in $[0,1]$ and all ideal sets f in $\mathcal{I}$. The <u>core</u> of v consists of the set of all μ in FA with

$$\mu(S) \geqq v(S)$$

for all S in $\mathcal{C}$, and

$$\mu(I) = v(I).$$

Superadditivity is a very well known condition in game theory; what it says is that disjoint coalitions do not lose by joining forces.

Homogeneity of degree 1 is a somewhat less known concept. An example of a set function that is homogeneous of degree 1 is any NA measure; the square or cube of such a measure is, however, not homogeneous of degree 1. More generally, let μ be a vector of measures in NA and let f be a real function, differentiable on the range R of μ, that is homogeneous of degree 1, i.e., such that $f(\alpha x) = \alpha f(x)$ for all α in $[0,1]$ and $x \in R$; then $f \circ \mu$ is in pNA and is homogeneous of degree 1. An example is $\sqrt{\mu^2 + \nu^2}$, where μ and ν are any two measures in NA^+. This, however, is not superadditive. But $-\sqrt{\mu^2 + \nu^2}$ <u>is</u> superadditive,**

*This follows from the continuity of v^* in the NA-topology (see (22.6)) and the denseness of $\mathcal{C}$ in $\mathcal{I}$ in that topology (Proposition 22.4).

**This follows from the triangle inequality for the euclidean norm.

and therefore satisfies all the conditions of Theorem F.

Another class of set functions v that are homogeneous of degree 1 is given by

(26.1) $v(S) = \max \{\int_S u(\underset{\sim}{x}(s),s)d\mu(s) : \int_S \underset{\sim}{x} d\mu = \int_S \underset{\sim}{a} d\mu\}$.

Here $\mu \in NA^+$; u is a real-valued function of two variables x and t, where x ranges over $[0,\infty)$ and t over I; $\underset{\sim}{a}$ is an integrable function from I to $[0,\infty)$; and the maximum is taken over all integrable functions $\underset{\sim}{x}$ from I to $[0,\infty)$ that satisfy the constraint (i.e., the statement after the colon).

These set-functions can be interpreted as models of productive economies, roughly as follows: $u(x,s)d\mu(s)$ is the amount of finished goods that producer s can produce from an amount x of raw materials and $\underset{\sim}{a}(s)d\mu(s)$ is the amount of raw materials available to s initially. Hence the total amount of raw materials initially available to a coalition S is $\int_S \underset{\sim}{a}(s)d\mu(s) = \int_S \underset{\sim}{a} d\mu$, and it may reallocate this amount among its members in any way it pleases; that is, if the members of S agree, then any $\underset{\sim}{x}$ that satisfies the constraints can be substituted for $\underset{\sim}{a}$. Therefore, if the maximum in (26.1) exists—i.e., if the supremum is finite and is attained—then the coalition S can reallocate its initial resources in such a way as to produce a total of v(S).

Whether the maximum in (26.1) indeed exists is a non-trivial question; it is treated, in a somewhat broader

context, in [A-P]. Even if it exists, it is not clear that the set-function v is in pNA; we plan to treat this question in Part V, again in a broader context. Here we will content ourselves with stating that if u is non-negative, uniformly bounded, and measurable in both variables simultaneously, and if for each fixed s, $u(\cdot,s)$ is non-decreasing in x and differentiable over all of $[0,\infty)$, then the max indeed exists and v is in pNA. However, the same conclusions can be reached under considerably wider conditions. In Part V we plan to treat this whole problem in much greater detail and generality, and also give alternative interpretations for the set functions v defined in (26.1).

To convince ourselves intuitively of the homogeneity of degree 1 of the set functions v defined in (26.1), let f be an ideal set. If g is any function integrable over I, it seems reasonable to define the "integral of g w.r.t. μ over the ideal set f" by

$$\int_f g\,d\mu = \int_I gf\,d\mu.$$

If in formula (26.1) we substitute the ideal set f for the ordinary set S, this definition of "integral over f" leads us to

$$v^*(f) = \max \{\int_I u(\underset{\sim}{x}(s),s)f(s)d\mu(s) \ : \ \int_I \underset{\sim}{x} f d\mu = \int_I \underset{\sim}{a} f d\mu\};$$

and $v^*(\alpha f) = \alpha v^*(f)$ would be a trivial consequence of this. Of course it must be proved that v^* is indeed given by the

above formula. This question too will be treated in Part V; here we only wanted to illustrate the notion of homogeneity of degree 1.

Finally, we would like to discuss the core. This concept is basic in game theory, and there is a large literature devoted to it.* In a game with a finite set N of players, the core consists of those payoff vectors** x with the property that no coalition S can assure itself more than it gets under x, while the all-player set N can in fact get x; this means that

$$\Sigma_{i \in S} x_i \geqq v(S) \tag{26.2}$$

for all $S \subset N$ and

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*See for example [B,G,K_3,R,S_6,S_7,S-S_1,S-S_2,Sc_2]; in particular, [K_3,R,Sc_2] are concerned with games with infinitely many players. All of these papers are about cores of "side-payment games," i.e., games defined by real-valued set functions, such as we have been treating here. Cores of non-side payment games have also been studied extensively; see for example [A_4,Bu,Sca], where the reader will also find the notion of "non-side payment game" defined. The notion of core is especially fruitful in connection with markets and other economic models (of which (26.1) is an example); in the side payment case see, for example, [S-S_1] and [S-S_2]. In the non-side payment case, the literature on the core of market and other economic games is very large indeed, especially when there are infinitely many players. We will therefore cite only [A_1] here, which is one of the relatively early papers on the subject. We should stress that the papers cited in this footnote were picked rather arbitrarily, and are far from constituting a complete bibliography on the core. This is not the place for such a bibliography, and many important papers on the subject were not cited.

**I.e., members of E^N, or functions from N to the reals. Intuitively, a payoff vector should be thought of as an assignment of a payoff to each player in the game.

(26.3) $$\Sigma_{i\epsilon N} x_i = v(N).$$

In games with infinitely many players, the idea of a payoff vector is most conveniently represented by a member μ of FA, i.e., a finitely additive measure. In this case $\mu(S)$ represents the total payoff to a coalition under μ; thus $\mu(S)$ corresponds to $\Sigma_{i\epsilon S} x_i$ in the finite case. It follows that the definition of core at the beginning of this section corresponds precisely to the classical definition (26.2) and (26.3) in the finite case.

Theorem F is proved in Sec. 27. We mention also Propositions 27.1, 27.8, and 27.12, and Remark 27.11, which have some independent interest.

27. PROOF OF THEOREM F

PROPOSITION 27.1. *Let v be a superadditive set function in pNA (or, more generally,* in pNA'). Then v^* is superadditive over the family $\mathcal{I}$ of ideal sets; that is,*

$$v^*(f+g) \geqq v^*(f) + v^*(g)$$

whenever $f, g, f+g \in \mathcal{I}$.

Proof. We will approximate to f and g, in the NA-topology, by *disjoint* ordinary sets; the result will then follow from the continuity of v^* in the NA-topology (see (22.6)) and the superadditivity of v.

Set $g_0 = f$, $g_1 = g$, $g_2 = f+g$. For given $\varepsilon > 0$, let μ_0, μ_1, μ_2 be three vector measures, and δ_0, δ_1, δ_2 three positive numbers, such that

$$\|\int(h-g_i)d\mu_i\| < \delta_i \implies |v^*(h) - v^*(g_i)| < \varepsilon$$

for $i = 0, 1, 2$. Let $\mu = (\mu_0, \mu_1, \mu_2)$. By Lemma 22.1, there are T_1 and T_2 in $\mathcal{C}$, with $T_2 \supset T_1$, such that for $i = 1, 2$,

$$\mu(T_i) = \int g_i d\mu;$$

if we set $T_0 = T_2 \setminus T_1$ and note that $g_0 = g_2 - g_1$, then this equation follows for $i = 0$ as well. In particular we have

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*See Sec. 22 at Proposition 22.10.

$$\int \chi_{T_i} d\mu_i = \mu_i(T_i) = \int g_i d\mu_i;$$

for i = 0, 1, 2. Hence

$$\|\int(\chi_{T_i} - g_i)d\mu_i\| = 0 < \delta_i,$$

and therefore

$$|v(T_i) - v^*(g_i)| = |v^*(\chi_{T_i}) - v^*(g_i)| < \varepsilon.$$

But from $T_2 = T_0 \cup T_1$ and $T_0 \cap T_1 = \emptyset$ it follows from the superadditivity of v that

$$v(T_2) \geqq v(T_0) + v(T_1).$$

Hence

$$\begin{aligned} v^*(f+g) = v^*(g_2) &\geqq v^*(g_0) + v^*(g_1) - 3\varepsilon \\ &= v^*(f) + v^*(g) - 3\varepsilon, \end{aligned}$$

and letting $\varepsilon \to 0$, we obtain the conclusion of Proposition 27.1.

In the remainder of this section, v will be a superadditive set function in pNA that is homogeneous of degree 1, φv will be its value, and v^* will be its extension.

<u>LEMMA 27.2</u>. <u>Let</u> $S \in \mathcal{C}$. <u>Then</u> $\partial v^*(t,S)$ <u>exists and is the same for all</u> t <u>in</u> (0,1).

<u>Proof</u>. By Theorem E, $\partial v^*(t,S)$ exists for almost all t in (0,1); let t_0 be a value of t for which it exists. Let $0 < t_1 < t_0$; then we have, by homogeneity,

$$\frac{v^*(t_1\chi_I + \tau\chi_S) - v^*(t_1\chi_I)}{\tau} = \frac{\frac{t_1}{t_0}\, v^*(t_0\chi_I + \frac{t_0}{t_1}\,\tau\chi_S) - \frac{t_1}{t_0}\, v^*(t_0\chi_I)}{\tau\chi_S}$$

$$= \frac{v^*(t_0\chi_I + \tau'\chi_S) - v^*(t_0\chi_I)}{t'\chi_S}$$

where $\tau' = \frac{t_0}{t_1}\tau$. When $\tau \to 0$, so does τ'. Hence when $\tau \to 0$, we have

$$\frac{v^*(t_1\chi_I + \tau\chi_S) - v^*(t_1\chi_I)}{\tau} \to \partial v^*(t_0,S);$$

thus $\partial v^*(t_1,S)$ exists and is equal to $\partial v^*(t_0,S)$. Since t_0 may be chosen arbitrarily close to 1, the proof of Lemma 27.2 is complete.

COROLLARY 27.3. *Let* $S \in \mathcal{C}$. *Then for all* $t \in (0,1)$,

$$(\varphi v)(S) = \partial v^*(t,S).$$

Proof. Follows from Theorem E and Lemma 27.2.

LEMMA 27.4. φv *is in the core of* v.

Proof. Let $S \in \mathcal{C}$. Fix an arbitrary t in $(0,1)$. By Proposition 27.1 we have, for all τ in $(0,1)$,

$$v(S) = v^*(\chi_S) = \frac{v^*(t\chi_I) + v^*(\tau\chi_S) - v^*(t\chi_I)}{\tau}$$

$$\leqq \frac{v^*(t\chi_I + \tau\chi_S) - v^*(t\chi_I)}{\tau}.$$

Letting $\tau \to 0+$ on the right and using Corollary 27.3, we deduce

$$v(S) \leqq \partial v^*(t,S) = (\varphi v)(S).$$

But $v(I) = (\varphi v)(I)$ is part of the definition of value (2.3); hence the proof of Lemma 27.4 is complete.

LEMMA 27.5. *Let* $\mu \in NA$ *be in the core of* v. *Then* $\mu = \varphi v$.

Proof. Fix an arbitrary t in $(0,1)$. Then from Proposition 27.1 and (21.3) we have

$$(27.6) \qquad v^*(t\chi_I) = tv(I) = t\mu(I) = \mu^*(t\chi_I).$$

But since μ is in the core of v, we have

$$\mu^*(f) \geqq v^*(f)$$

for all $f \in \mathcal{J}$; the proof of this is similar to that of Proposition 27.1. In particular, therefore, for $\tau > 0$ sufficiently small, we have, for any $S \in \mathcal{C}$,

$$(27.7) \qquad \mu^*(t\chi_I) + \tau\mu(S) = \mu^*(t\chi_I + \tau\chi_S)$$
$$\geqq v^*(t\chi_I + \tau\chi_S).$$

Combining (27.6) with (27.7), we get

$$\mu(S) = \frac{\mu^*(t\chi_I) + \tau\mu(S) - \mu^*(t\chi_I)}{\tau}$$

$$\geqq \frac{v^*(t\chi_I + \tau\mu_S) - v^*(t\chi_I)}{\tau} .$$

Letting $\tau \to 0+$, we deduce from Corollary 27.3 that

$$\mu(S) \geqq \varphi v^*(t,S) = (\varphi v)(S).$$

Since S was arbitrary, we have also

$$\mu(I\backslash S) \geqq (\varphi v)(I\backslash S).$$

Using $\mu'(I) = v(I) = (\varphi v)(I)$ we obtain $\mu(S) = (\varphi v)(S)$, and the proof of Lemma 27.5 is complete.

It remains to prove that the core of v contains only NA measures. In fact, we have the more general

PROPOSITION 27.8. *If* $w \in AC$, *then every member of the core** *of* w *is in* NA.

Proof. Related results have been obtained by Schmeidler and by Rosenmüller.** Our proof follows Schmeidler's ideas closely. Let $\nu \in NA^+$ be such that $w \ll \nu$, and let μ be the core of w. Let $T_1, T_2, \ldots$ be a sequence of sets in $\mathcal{C}$; we claim

$$\nu(T_i) \to 0 \text{ implies } \mu(T_i) \to 0. \tag{27.9}$$

Indeed, from $\nu(T_i) \to 0$ and $w \ll \nu$ it follows that

$$w(T_i) \to 0 \text{ and } w(I\backslash T_i) \to W(I). \tag{27.10}$$

Using $\mu(T_i) \geqq w(T_i)$, $\mu(I\backslash T_i) \geqq w(I\backslash T_i)$, and $\mu(I) = w(I)$, we deduce

*Of course, there is no assertion here that the core is non-empty.

**See [Sc_2], Lemmas A and C, and [R], Theorem 1.2 and Corollary 2.4. Both authors assume $v(S) \geqq 0$ for all S, and this simplifies matters somewhat.

$$\liminf \mu(T_i) \geqq \liminf w(T_i) = 0$$

and

$$\limsup \mu(T_i) = \mu(I) - \liminf \mu(I \setminus T_i)$$
$$\leqq \mu(I) \leqq \liminf w(I \setminus T_i) = \mu(I) - w(I) = 0;$$

thus $\lim \mu(T_i)$ exists and $= 0$, and (27.9) is proved.

Now let $S = \cup_{j=1}^{\infty} S_j$, where the S_j are disjoint. If we set $T_i = S \setminus \cup_{j=1}^{i} S_j$, then the T_i obey the hypotheses of (27.9); hence $\lim \mu(T_i) = 0$, i.e.,

$$\mu(S) = \lim \Sigma_{j=1}^{i} \mu(S_j) = \Sigma_{j=1}^{\infty} \mu(S_j).$$

Thus μ is completely additive.

To show that μ is non-atomic, let $s \in I$, and in (27.9), let $T_i = \{s\}$ for all i. From $v \in NA$ we get $\lim v(T_i) = v(\{s\}) = 0$. Hence $0 = \lim \mu(T_i) = \mu(\{s\})$, and the proof of Proposition 27.8 is complete.

Theorem F follows immediately from Lemmas 27.4 and 27.5, Proposition 27.8, and the fact that $pNA \subset AC$ (Corollary 5.3).

<u>Remark 27.11</u>. It is perhaps worth noting that Proposition 27.8 continues to hold if we assume $w \in bv'NA$ rather than $w \in AC$. Indeed, for the proof it is sufficient that there exist an NA^+ measure v such that $v(T_i) \to 0$ implies (27.10); and for $w \in bv'NA$ there does indeed exist such a measure.

The following proposition will be useful in applications of Theorem F.

PROPOSITION 27.12. The set of superadditive members of pNA that are homogeneous of degree 1 is closed in BV.

Proof. pNA is closed by definition. The closedness of the set of superadditive set functions follows from the continuity, for each fixed $S \in \mathcal{C}$, of the mapping $v \to v(S)$ from BV to the reals. To show that the space of set functions that are homogeneous of degree 1 is closed, note that $v \to v^*$ is continuous (see (22.7)), and that for each fixed $f \in \mathcal{J}$, $v^* \to v^*(f)$ is continuous. Therefore for each $\alpha \in [0,1]$, the mapping $v \to v^*(\alpha f) - \alpha v^*(f)$ is continuous, and Proposition 27.12 is proved.

We close this section with yet another*

Alternative Proof for Example 5.8. If $v \in$ pNA, then so is $-v$; then by (22.12) we have

$$-v^*(\alpha f) = -\left|\int \alpha f d\mu\right| = \alpha\left|\int f d\mu\right| = \alpha(-v^*(f)),$$

and hence $-v$ is homogeneous of degree 1. Furthermore $-v$ is clearly superadditive. But μ and $-\mu$ are both in the core of v, in contradiction to Theorem F. So $v \notin$ pNA, as was to be proved.

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*Proofs were previously given in Secs. 5, 17, and 22.

REFERENCES

[I] Aumann, R. J., and L. S. Shapley, Values of Non-Atomic Games, Part I: The Axiomatic Approach, The Rand Corporation, RM-5468-PR, November 1968; also The Hebrew University of Jerusalem, Research Program in Game Theory and Mathematical Economics, RM 42, December 1968.

[II] ______, Values of Non-Atomic Games, Part II: The Random Order Approach, The Rand Corporation, RM-5842-PR, July 1969.

[III] ______, Values of Non-Atomic Games, Part III: Values and Derivatives, The Rand Corporation, RM-6216, February 1970; also the Hebrew University of Jerusalem, Research Program in Game Theory and Mathematical Economics, RM-49, October 1969.

[A_1] Aumann, R. J., "Markets with a continuum of traders," Econometrica 32 (1964), pp. 39-50.

[A_4] ______, "The core of a cooperative game without side payments," Trans. Amer. Math. Soc. 98 (1961), pp. 539-552.

[A-P] Aumann, R. J., and M. Perles, "A variational problem arising in economics," J. Math. Anal. Appl. 11 (1965), pp. 488-503.

[B] Bondareva, O. N., "Some applications of linear programming methods to the theory of cooperative games" (in Russian), Prob. Kibernet. 10 (1963) pp. 119-139.

[Bu] Burger, E., "Bemerkungen zum Aumannschen Core-Theorem," Z. Warsch. 3 (1964), pp. 148-153.

[G] Gillies, D. B., "Solutions to general non-zero sum games," in Contributions to the Theory of Games IV, Princeton University Press, Princeton, 1959, pp. 47-85.

[K_3] Kannai, Y., "Countably additive measures in cores of games," J. Math. Anal. Appl. 27 (1969), p. 227-240.

[R] Rosenmüller, On Core and Value, School of Mathematics, Institute of Technology, University of Minnesota (Mimeographed), 1969.

[S_6] Shapley, L. S., Notes on n-Person Games VII: Cores of Convex Games, The Rand Corporation, RM-4571-PR, October 1965.

[S_7] ______, "On balanced sets and cores," Naval Res. Logist. Quart. 14 (1967), pp. 453-460.

[S-S_1] Shapley, L. S., and M. Shubik, "Quasi-cores in a monetary economy with nonconvex preferences," Econometrica 34 (1966), pp. 805-827.

[S-S_2] ______, "On market games," J. Econ. Theory 1 (1969), pp. 9-25.

[Sc_2] Schmeidler, D., On Balanced Games with Infinitely Many Players, The Hebrew University of Jerusalem, Research Program in Game Theory and Mathematical Economics, RM 28, June 1967.

[Sca] Scarf, H. "The core of an n-person game," Econometrica 35 (1967), pp. 50-69.